THERMAL CONDUCTIVITY OF SOLIDS

Larry Glasgow
Department of Chemical Engineering
Kansas State University
Manhattan, KS 66505-5102

INTRODUCTION

The title of this laboratory experiment is misleading; the procedure we are about to employ is a very poor one for the determination of *k* of unknown materials. However, the experiment is an extremely useful introduction to unsteady-state heat transfer and the lessons we learn here are directly applicable to other, practical heat transfer situations. This procedure is carried out with the apparatus shown in Figure 1. The sample or circulation chamber is on the left, and the heated bath on the right. The control unit for the electric immersion heater is mounted on the front (top) of the heated bath; the pump motor can be seen immediately behind it.

Figure 1. Thermal conductivity apparatus, with sample chamber (left), hot water bath, immersion heater control, and pump. One of the polycarbonate samples is standing on end in the middle.

We are going to examine the behavior of the cylindrical samples; several of these are 6 inches long and 1 inch in diameter. The polycarbonate and phosphor bronze specimens are pictured below in Figure 2; you can easily identify the mounting stems and the leads from the copper-constantan thermocouples, which are located on the sample centerline.

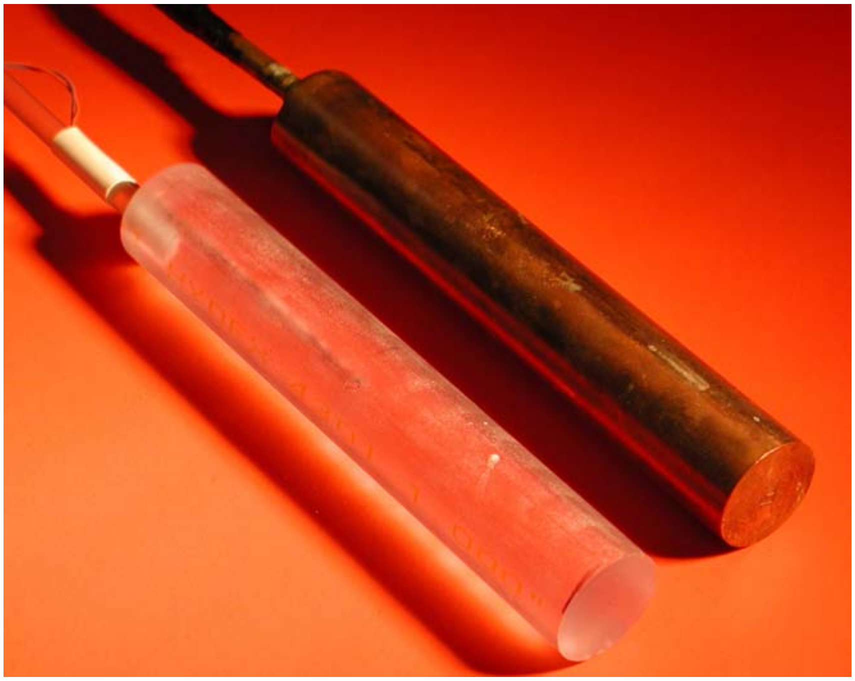

Figure 2. Polycarbonate and phosphor bronze (cylinder "F") test specimens.

Our procedure is very simple; the thermocouple leads are connected to a recording device and at t=0, the sample is plunged into the heated water in the test chamber.

AN ANALYTIC SOLUTION

Consider a solid cylindrical billet, at some uniform initial temperature (say 3 °C); at t=0 this sample is plunged into a heated bath maintained at about 65 to 70 °C. By recording the emf produced by a copper-constantan thermocouple on the cylinder centerline, we can obtain a record of the approach of the sample's temperature to that of the heated bath. Clearly in the interior of the solid sample, heat transfer occurs solely by conduction; therefore, the appropriate form of equation B.9-2 (page 850 in *Transport Phenomena*) is:

$$\rho C_p \frac{\partial T}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] \qquad (1)$$

If the cylinder is infinitely long, or practically speaking, if *L/D* is sufficiently large, then axial conduction can be neglected. You should carefully consider the circumstances under which this is a reasonable assumption. How might you go about assessing this simplification, quantitatively? It proves useful to employ a dimensionless temperature, defined by:

$$\theta = \frac{T - T_b}{T_i - T_b}, \qquad (2)$$

where T_b is the temperature of the heated bath and T_i is the initial temperature of the specimen. By this definition θ=1 initially, and $\theta \to 0$ as $t \to \infty$. We introduce θ into eq. (1), and divide by ρC_p. The result is:

$$\frac{\partial \theta}{\partial t} = \alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) \right] \qquad (3)$$

We now illustrate a method by which the analytic solution of eq. (3) can be determined; this technique is often referred to as the product method, or separation of variables. We *postulate* that a solution can be found of the form:

$$\theta = f(r)g(t), \qquad (4)$$

where *f* is a function solely of *r* and *g* is a function solely of *t*. Consider the consequences of introducing (4) into (3):

$$fg' = \alpha (gf'' + \frac{1}{r} gf') \qquad (5)$$

Prove that this is correct. Divide eq. (5) by the product, $f \cdot g$. The result is:

$$\frac{g'}{\alpha g} = \frac{f'' + \frac{1}{r} f'}{f} \qquad (6)$$

Note that the left-hand side is a function *only* of time. The right-hand side is a function *only* of radial position. Yet, they are equal. Obviously, both sides of (6) must be equal to a constant, which we write as $-\lambda^2$. The rationale for this choice will become apparent momentarily. It is evident that we now have two *ordinary* differential equations:

$$\frac{dg}{g} = -\alpha \lambda^2 dt \quad \text{and} \quad \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \lambda^2 f = 0 \qquad (7a, 7b)$$

Show that the solution to (7a) is: $g = C_1 \exp(-\alpha\lambda^2 t)$. You should also recognize that (7b) is a form of Bessel's differential equation; see Mickley, Sherwood, and Reed, Applied Mathematics in Chemical Engineering, McGraw-Hill, 1957). The solution for (7b) has the form:

$$f = AJ_0(\lambda r) + BY_0(\lambda r),$$

where J_0 and Y_0 are zero-order Bessel functions of the first and second kind, respectively. According to our hypothesis,

$$\theta = C_1 \exp(-\alpha\lambda^2 t)[AJ_0(\lambda r) + BY_0(\lambda r)]. \qquad (8)$$

You may wish to verify that (8) is in fact a solution for eq. (3). We have two boundary conditions that must be satisfied; first, at $r=0$, θ must be finite. Since $Y_0(0)=-\infty$, we set B=0. Consider the boundary condition to be applied at $r=R$; if the cylinder surface attains the bath temperature *very rapidly*, then at $r=R$, $\theta=0$. Therefore, $J_0(\lambda R)=0$. J_0 has infinitely many zeros, irregularly spaced. We have no reason to believe that at fixed time and radial position, any single one of the infinite number of possible values of λ would result in solution. Therefore, we use superposition to rewrite (8):

$$\theta = \sum_{n=1}^{\infty} A_n \exp(-\alpha\lambda_n{}^2 t) J_0(\lambda_n r) \qquad (9)$$

Whether or not the boundary condition at $r=R$ is appropriate depends upon the relative rates of heat transfer on the two sides of the fluid-solid interface. If the cylindrical sample has a (relatively) large thermal conductivity, then heat flow to the interior will occur rapidly and preclude use of this boundary condition. In fact, this will be the general situation with the metallic samples we examine. For these cases, a Robin's type boundary condition must be employed at $r=R$ where the thermal energy fluxes are equated on either side of the interface. We accomplish this by using Fourier's law and Newton's "law" of cooling,

$$-k\frac{dT}{dr}_{r=R} = h(T_{r=R} - T_b). \qquad (10)$$

After introducing our dimensionless temperature and performing the indicated differentiation (term-by-term), this boundary condition can be rewritten as:

$$\lambda_n R J_1(\lambda_n R) = \frac{hR}{k} J_0(\lambda_n R) \qquad (11)$$

This transcendental equation occurs frequently in mathematical physics; roots are compiled in many places, including Carslaw and Jaeger (Conduction of Heat in Solids) in Appendix IV, Table III. Look carefully at the quotient, hR/k. *It is not the Nusselt number; it is the Biot modulus.* Make sure you know the difference. Now, suppose that hR/k=1.5; in this case the first six roots for $\lambda_n R$ are:

1.4569, 4.1902, 7.2233, 10.3188, 13.4353, and 16.5612.

The use of (10) as a boundary condition poses a serious problem; we have no *a priori* means of determining h. Thus, we have introduced *another unknown parameter* into an experimental procedure that was intended to provide a means for estimating k (or the thermal diffusivity, α). The deficiency of this experiment is now clear: *The determination of thermal conductivity will only be possible if the main resistance to heat transfer is in the material—not in the fluid phase*. Before we attempt to resolve this difficulty, we need to finish our analytic solution. This means choosing values for the leading coefficients (the A_n's) that cause our series to converge to the desired solution. Note that we have applied two boundary conditions--we now employ the initial condition: For all time up to t=0, the sample temperature is a uniform, T_i, such that θ=1. Therefore, we rewrite (9) as

$$1 = \sum A_n J_0(\lambda_n r)\,. \qquad (12)$$

We now take advantage of the orthogonality of Bessel functions; in particular, we're going to make use of the following type of relationship:

$$0 = \int_0^R r J_0(\lambda_n r) J_0(\lambda_m r)\,dr \qquad \text{for } n \neq m. \qquad (13)$$

Thus, in principle, we multiply both sides of (12) by $rJ_0(\lambda_n r)dr$ and integrate from 0 to R to determine the unknown coefficients. This is illustrated in Powers (1979) on pages 220 and 221. *Please make note of the fact that (for n=m) eq. (13) will result in*

$$A_n = \frac{2}{\lambda_n R J_1(\lambda_n R)} \qquad (14)$$

<u>*only for the case*</u> *in which the* λ_n*'s are the roots of* $J_0(\lambda_n R) = 0$; i.e., for the <u>non-metal samples</u>. Our situation with the metallic billets is more complicated since the separation constants have come from the Robin's type boundary condition, eq. (11). It is not a straightforward exercise to show:

$$A_n = \frac{2\lambda_n R J_1(\lambda_n R)}{\left(\frac{h^2R^2}{k^2} + \lambda_n{}^2 R^2\right) J_0{}^2(\lambda_n R)} \qquad (15)$$

One other important question remains: How fast does the series, eq. (9), converge? If more than 3 or 4 terms are required, the analytic solution will not be very useful. Note that if α and/or t are large, the exponential factor will certainly be dominant. Let's explore series convergence for a specific case; consider the phosphor bronze cylinder (sample F):

PHOSPHOR BRONZE SPECIMEN

L=15.24 cm D=2.54 cm

ρ=8.86 g/cm^3 C_p=0.09 cal/(g °C)

k=0.165 cal/(cm^2 s °C)/cm

α=0.2074 cm^2/s

Now we take $\frac{hR}{k} = 0.15$; we'll determine whether or not this is an appropriate choice later. Using tabulated roots for eq. (11), we find that:

n	λ_n	$\lambda_n R$	A_n
1	0.42	0.5376	1.0356
2	3.05	3.8706	-0.0492
3	5.54	7.0369	+0.0202
4	8.02	10.1882	?
5	10.50	13.3349	?
6	12.98	16.4797	?

You may want to try to complete this table as an exercise. Now look at the centerline temperature of the phosphor bronze specimen 5 s after immersion in the heated bath:

First term of infinite series: 0.8625

Second term: $-3.221\text{x}10^{-6}$

This is highly desirable behavior in an infinite series solution!

OBTAINING A NUMERICAL SOLUTION

The limitations of the infinite series solution are apparent. If t is small, <u>many</u> terms may be required for convergence. Fortunately, we have a simple (yet powerful) technique that will allow us to arrive at a solution of the partial differential equation for the case in which the thermal diffusivity is known. Since we have already compiled the required information for phosphor bronze, we'll treat that case as our example. Our starting point is eq. (1) and once again, we'll neglect axial conduction in the cylindrical sample.

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \qquad (16)$$

Let the indices, *i* and *j*, represent radial position and time, respectively. We now write a finite difference representation of this equation:

$$T_{i,j+1} = \alpha \Delta t \left[\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta r)^2} + \frac{1}{(i-1)\Delta r} \frac{T_{i+1,j} - T_{i,j}}{\Delta r} \right] + T_{i,j} \qquad (17)$$

Notice how this equation allows us to compute the temperature on the new time-step row ($j+1$), using only known, old temperatures. This is an *explicit* algorithm for solution of the partial differential equation. Please note that we cannot let the quotient, $\alpha\Delta t/(\Delta r)^2$, assume any arbitrary value. This parameter must be kept smaller than 0.5 for numerical stability; see G. D. Smith, Numerical Solution of Partial Differential Equations for elaboration. For purposes of our experiment, I have divided the radius of the cylindrical billet into 50 radial segments, which means that Δr=0.0254 cm. Accordingly, the index i will assume values from 1 and 51, inclusive. A simple BASIC code was developed for this problem, and a listing is provided immediately below in Figure 3. An executable version of the program will be made available to you.

```
10 REM *** EXPLICIT COMPUTATION OF TEMPERATURE PROFILE IN SOLID CYLINDER
20 REM     INSTANTANEOUSLY IMMERSED IN A HEATED BATH.  PROGRAM WRITTEN
30 REM     FOR CHE LAB 1 EXPERIMENT, "THERMAL CONDUCTIVITIES OF SOLIDS"
35 REM     PROGRAM FOR 1-INCH ALUMINUM CYLINDER BY L. GLASGOW
40 OPTION BASE 1
50 DIM T(51,2)
51 cls
52 print "*************************************************************"
53 print "**  THIS PROGRAM WAS CREATED FOR THE THERMAL CONDUCTIVITY    **"
54 PRINT "**  EXPERIMENT.  IT CALCULATES THE TEMPERATURE DISTRIBUTION  **"
55 PRINT "**  IN A ONE INCH DIAMETER CYLINDRICAL METAL BILLET THAT IS  **"
56 PRINT "**  PLUNGED INTO A HEATED WATER BATH.  THE PROGRAM EMPLOYS   **"
57 PRINT "**  CGS UNITS.  THE DEMO VALUES PROVIDED WERE OBTAINED FOR   **"
58 PRINT "**  THE ALUMINUM SPECIMEN.  RUN THIS CASE FIRST.  NOTE THAT  **"
59 PRINT "**  THE PROGRAM RUNS ONLY UNTIL CENTER TEMPERATURE ATTAINS   **"
60 PRINT "**  90% OF THE ULTIMATE VALUE.  L. GLASGOW, CHE 522.         **"
61 PRINT "**  THE DATA ARE WRITTEN TO THE SCREEN AND TO DRIVE A:       **"
62 PRINT "*************************************************************"
63 OPEN "A:TCENTER2.PRN" FOR OUTPUT AS #1
70 DR=1.27/50:dt=0.0002
71 PRINT "  "
72 PRINT "Specify the heated bath temperature, usually about 75 C"
73 input tbath
75 T=0
76 print "  "
77 PRINT "Select an appropriate heat transfer coefficient (.034)"
79 print "Familiar AE units can be obtained by dividing by 0.0001356"
80 print "So, h=.034/.0001356=250.7 Btu/hr ft^2 F":input h
81 print "  "
82 print "Specify the thermal diffusivity of sample (0.86 cm^2/s)"
83 input alf
84 print "  "
86 print "Select the thermal conductivity of sample (0.48 cal/cm s C)"
87 input kcon
90 rem
95 KNT=1
100 REM *** INITIALIZATION
101 print "  "
102 print "Now choose the ice water bath temperature, normally 3 C"
103 input tcold
110 FOR I=1 TO 51
120 T(I,1)=tcold
130 NEXT I
140 REM *** CALCULATE OUTER SURFACE TEMPERATURE WITH BOUNDARY CONDITION
150 T(51,1)=(-1/(12*DR)*(3*T(47,1)-16*T(48,1)+36*T(49,1)-
48*T(50,1))+H/KCON*TBATH)/(25/(12*DR)+H/KCON)
160 REM *** EXPLICIT COMPUTATION
170 FOR I=2 TO 50
180 T(I,2)=ALF*DT*((T(I+1,1)-2*T(I,1)+T(I-1,1))/DR^2+1/((I-.999)*DR)*(T(I+1,1)-
T(I,1))/DR)+T(I,1)
190 NEXT I
200 REM *** FLOP TIME VALUES
210 FOR I=2 TO 50
220 T(I,1)=T(I,2)
230 NEXT I
240 REM *** APPLY BOUNDARY CONDITION AT CENTERLINE
250 T(1,1)=1/25*(48*T(2,1)-36*T(3,1)+16*T(4,1)-3*T(5,1))
260 T=T+DT:KNT=KNT+1
265 IF KNT<400 THEN 140 ELSE 270
270 PRINT "TIME=";T:KNT=1
272 WRITE#1,T,T(1,1)
280 PRINT "CENTERLINE TEMPERATURE=";T(1,1)
285 PRINT "SURFACE TEMPERATURE=";T(51,1)
290 IF T(1,1)<.9*TBATH THEN 140 ELSE 300
300 CLOSE:END
```

We're going to use this program to determine the "best" value for the heat transfer coefficient, *h*, by comparison with experimental data for the phosphor bronze cylinder. You should be aware of the fact that when water is used for heating or cooling in agitated tanks, *h* typically is on the order of 50 to perhaps 3000 Btu/(hr ft^2 °F). In the cgs system (which is extremely appropriate for our experiment), this range of values corresponds to 0.0068 through 0.407 cal/(s cm^2 °C). In this experiment, experience has shown the the heat transfer coefficient is almost always between 100 and 225 Btu/(hr ft^2 °F). We shall test this shortly, but we need to actually perform the experiment.

CARRYING OUT THE EXPERIMENT

Begin with the phosphor bronze sample immersed in ice water, allowing it to attain a uniform temperature of around 3 °C. This period of equilibration should be about 5 minutes. See that the heated bath is at a temperature of at least 65 °C. Using the acrylic plastic cover to secure the billet, plunge the specimen into the heated bath at t=0; make sure that the recording device (Keithley 2700) is started simultaneously. We obtain a record of centerline temperature as a function of time (actually thermocouple emf). Note that the phosphor bronze sample will come to equilibrium within ~100 s of immersion. Repeat this process for each of the metallic samples; depending upon your time constraints, you may want to run each sample twice.

The nonmetal cylinders present some different challenges. For example, consider the acetal polymer cylinder (identified by the letter, "B"); according to Perry's Chemical Engineers' Handbook the density, heat capacity, and thermal conductivity are:

ρ=1.425 g/cm^3, C_p=0.35 cal/(g °C), k=0.000537 cal/(s cm^2 °C).

As a result, the thermal diffusivity, α, of acetal is only about 0.00108 cm^2/s. Compare this to the thermal diffusivity for phosphor bronze cited previously. The conduction of thermal energy in the acetal polymer specimen is going to be *very much slower*. We need to take this into account when cooling or heating the nonmetal samples. Here is an abbreviated table to help you get a feel for the range of thermal diffusivities:

Material	Thermal diffusivity, α (cm^2/s)
silver	1.71
copper	1.14
aluminum	0.86 (0.835 according to Perry's)
brass	0.33
cast iron	0.12
monel	0.053
stainless steel (304)	0.040
glass	0.0058
wood (spruce, across grain)	0.0024
acrylic plastic	0.0012

We're employing copper-constantan thermocouples, so we need to know how to convert emf to temperature. Consulting Lange's (Handbook of Chemistry), we find the calibration table for copper-constantan thermocouples *with the reference or "cold" junction maintained at* 0 °C:

Temperature, °C	TC output, mV
0	0.000
10	0.389
20	0.787
30	1.194
40	1.610
50	2.035
60	2.467
70	2.908

How well can these data be represented by a straight line? You can appreciate why it is so important for us to monitor both the ice water and heated bath temperatures with accurate thermometers.

What do the data look like? A composite of three experimental runs with the phosphor bronze cylinder appears below as Figure 4. Note that the centerline temperature does not begin to increase instantaneously. You will also observe that the bath temperature has nearly been attained by the center of the sample at about 70 s following immersion. We can expect some of the nonmetals to react much more slowly! In fact, for these cases (acrylic or Plexiglas, acetal, polyethylene, polycarbonate) we can make immediate use of Figure 12.1-2 on page 378 of *Transport Phenomena* or Figure 7 in the Appendix.

EXPERIMENTAL STRATEGY

Begin the experiment with the phosphor bronze sample; you should obtain data similar to those shown in Figure 4. *Our plan is to use these data (in conjunction with the numerical solution of the model) to determine the "best" value of the heat transfer coefficient.* Figure 5 illustrates model computations for two values of h: 100 and 250 Btu/(hr ft^2 °F). Compare these curves with the data appearing in Figure 4. It is immediately apparent that the most suitable value for the heat transfer coefficient is approximately 125 to 150 Btu/(hr ft^2 °F); in the cgs system, this range corresponds to 0.0169 to 0.0203 cal/(s cm^2 °C).

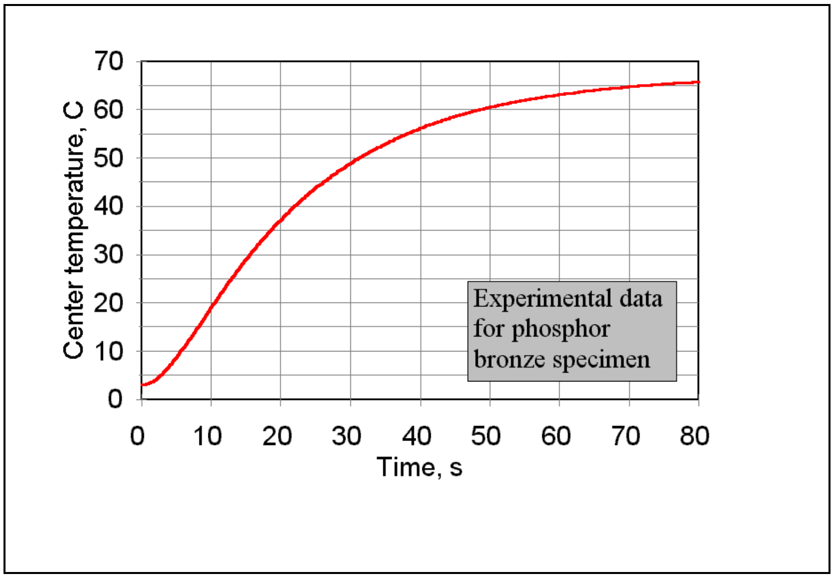

Figure 4. Experimental data obtained with the phosphor bronze cylinder. The initial temperature was 3°C and the final (heated bath) temperature was 67°C. Notice that 90% of the temperature change is accomplished in just 50 s.

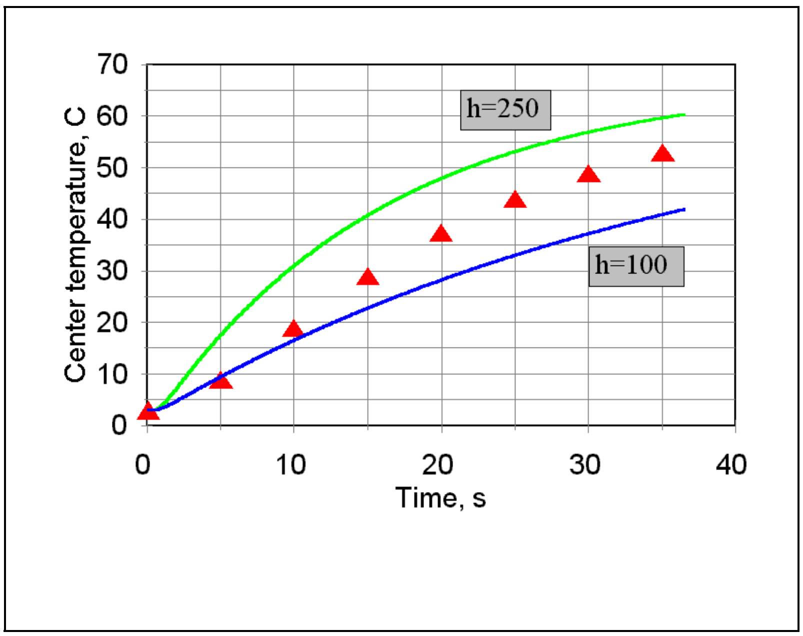

Figure 5. Comparison of data for phosphor bronze sample with model computations using two values of *h*: 100 and 250 Btu/(hr ft^2 °F). The markers for the experimental data are triangles.

Now that you have a reliable value for *h*, can you think of a procedure that would allow you to estimate the thermal diffusivities for "unknown" samples? Examine eq. (9). Suppose the infinite series solution converges so rapidly that only the first term had to be retained; under those circumstances,

$$\theta \approx A_n \exp(-\alpha\lambda_n{}^2 t) \qquad \text{since} \qquad J_0(0) = 1.0\,. \tag{18}$$

Consequently,

$$\ln(\theta) \approx \ln(A_n) - \alpha\lambda_n{}^2 t \tag{19}$$

In principle, a plot of $ln\theta$ as a function of time, t, should yield slope and intercept; for the phosphor bronze case illustrated in Figure 6 below, you'll note that the slope is about -0.0365 (verify!) and the intercept is about 1.31. Make absolutely certain that you are comfortable with slope determination on both logarithmic and semi-logarithmic plots. Let's turn back to eq. (11) and arbitrarily select hR/k=1; according to the appropriate table in Carslaw and Jaeger, $\lambda_1 R$=1.2558. Therefore, $\alpha \cong 0.0365/(0.9888)^2 = 0.037$ cm^2/s. Now we combine equations (11) and (15) to produce:

$$A_n = \frac{2hR/k}{(h^2R^2/k^2 + \lambda_n{}^2R^2)J_0(\lambda_n R)} \tag{20}$$

Show that: $18.4 = \dfrac{1}{0.00054/k + 1.577k}$.

This <u>does not yield</u> a physically realizable value for the thermal conductivity, k. Our selection of the parametric value, hR/k=1 was a poor one! Try 0.15 instead. What value do you obtain (for k) this time? You should get about 0.114 cal/(cm s °C), which is about 30% too low. It is obvious that this method is greatly dependent upon construction of the straight line in Figure 6.
We need a better procedure for the determination of thermal conductivity; fortunately, the numerical solution of the partial differential equation provides just that. Use the program (with your value of the heat transfer coeffcient) to determine the thermal diffusivity, α, for several of the "unknown" specimens. Do this by adjusting the value of the diffusivity until the computed temperature history closely approximates your experimental results. What are the major factors affecting your level of confidence in the resulting estimates for k? Could you devise a better experimental procedure for determination of thermal conductivity of unknown materials? How would your test procedure vary from the one employed in this experiment?

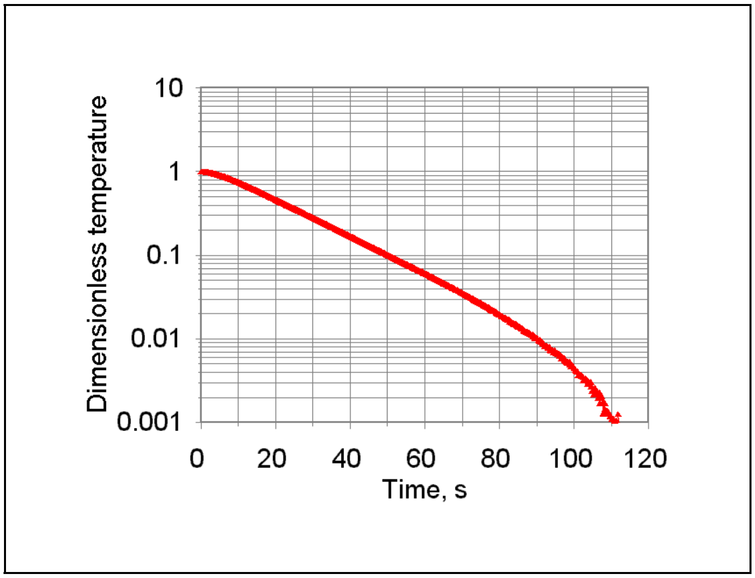

Figure 6. Dimensionless temperature as a function of time for the phosphor bronze specimen. Note that the central portion can be approximately represented with a straight line.

LITERATURE CITED

Bird, R. B., Stewart, W. E., and E. N. Lightfoot. Transport Phenomena, 2nd Edition, Wiley, New York (2002).

Carslaw, H. S. and J. C. Jaeger. Conduction of Heat in Solids, Oxford University Press, Oxford (1959).

Lange, N. A., Editor. Handbook of Chemistry, McGraw-Hill, New York (1961).

Mickley, H. S., Sherwood, T. K., and C. E. Reed. Applied Mathematics in Chemical Engineering, McGraw-Hill, New York (1957).

Perry, R. H., Chilton, C. H., and S. D. Kirkpatrick. Chemical Engineers' Handbook, Fourth Edition, McGraw-Hill, New York (1963).

Powers, D. L. Boundary Value Problems, Second Edition, Academic Press, New York (1979).

Smith, G. D. Numerical Solution of Partial Differential Equations, Oxford University Press, Oxford (1975).

APPENDIX

Solution for the case in which the surface of the cylinder instantaneously attains the bath temperature.

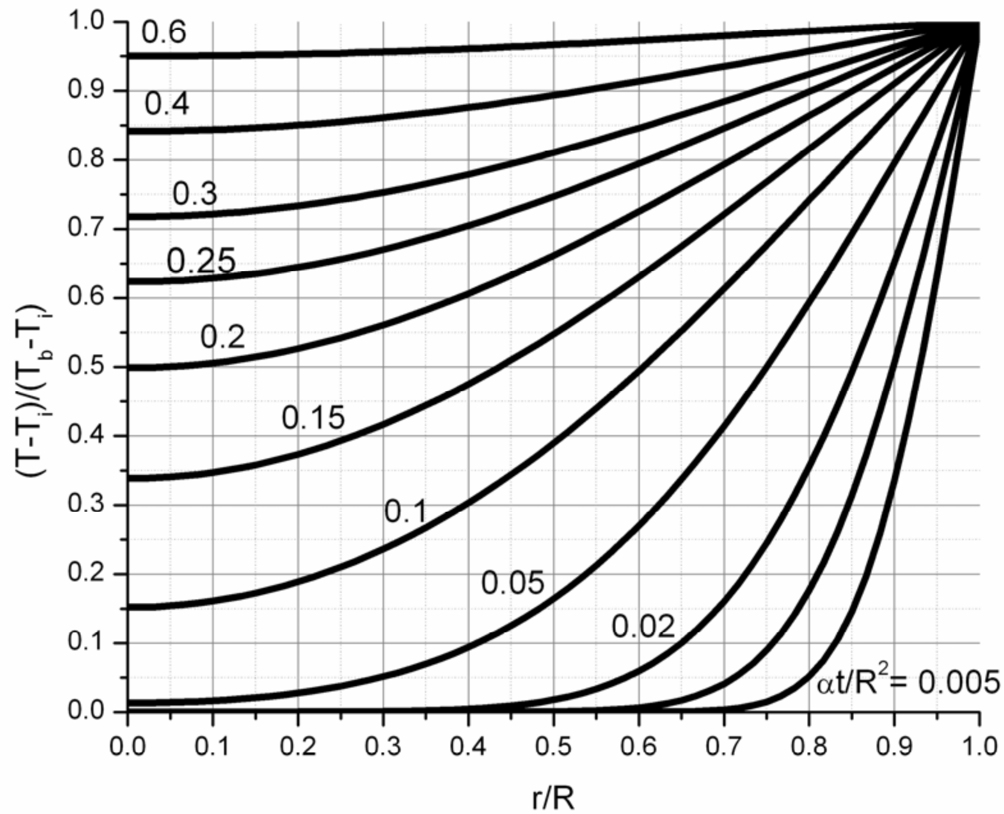

Figure 7. Temperature distributions for transient conduction in a long cylinder. The initial temperature of the material is T_i; at t=0, the outer surface (r=R) is instantaneously heated to T_b. The curves represent values of $\alpha t/R^2$ ranging from 0.005 to 0.60 and the center of the cylinder corresponds to the left-hand side of the figure. The data appearing in this figure were computed numerically.